

Proceedings of the Second Bornö Symposium

on

Geophysical Fluid Mechanics

edited by

Pierre Welander



Some of the participants in discussion at the coffee table.

(left to right): Veronis, Howard, Spiegel, Hide, Rooth.

(standing): Malkus.

II. List of participants

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Dr. Raymond Hide, Massachusetts Institute of Technology, U.S.A.

Dr. Louis N. Howard, Massachusetts Institute of Technology, U.S.A.

Dr. Willem V.R. Malkus, University of California, U.S.A.

Dr. Joseph Pedlosky, Massachusetts Institute of Technology, U.S.A.

Dr. Allan Robinson, Harvard University, U.S.A.

Mr. Claes Rooth, Woods Hole Oceanographic Institution, U.S.A.

Dr. Edward A. Spiegel, New York University, U.S.A.

Dr. Bernard St. Guily, Museum d'Histoire Naturelle, France.

Dr. Melvin Stern, Woods Hole Oceanographic Institution, U.S.A.

Dr. George Veronis, Massachusetts Institute of Technology, U.S.A.

Mr. Sten Gösta Walin, Stockholm University, Sweden.

Dr. Pierre Welander, Stockholm University, Sweden and
Woods Hole Oceanographic Institution, U.S.A.

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I. Introduction

It is felt that the large scientific meetings may not give the participants opportunity to present their works in such detail as many colleagues would like to see, and that the discussion of the contributions is often held back by fear of the chairman's clock and the mass of people present. It is then useful to resort to small symposia, preferably of international type. A few years ago the idea came up to assemble a group of scientists working on hydrodynamical problems in geophysics and oceanography, to hold such a symposium in Sweden. This idea materialized eventually, and in June 1962 the first symposium was held at the Bornö Field Station with nine participants from the United States and from Sweden. This station is located on the Stora Bornö, a small island in the Gullmarsfjord on the Swedish west coast. The station, which belongs to the Swedish Fishery Board, has accommodations for about ten scientists, and meals are also provided by a "husmor".

On this symposium only two or three talks were given each day, and in practice the speaker could go on as long as he had the energy and could keep the audience interested. In the evenings the participants could continue the discussions, or take some free time to go swimming or sauna-bathing, climbing rocks or making excursions by the boat. It was felt that by meeting on a small island the scientists could create their own little world for a

couple of weeks, where they could concentrate their thoughts on the scientific problems without being disturbed by all the impressions from "modern" surroundings. The participants seemed also very positive for this type of symposium, and it was argued that the experiment should be repeated two years later, and eventually become regular.

The second Bornö Symposium was planned accordingly and took place during June 8 - 18, 1964. The number of participants were now fourteen, and there were difficulties accommodating them all at the station. The fishing boat "Anna-Greta" was therefore hired and placed at the station, to give a few extra beds. She was also used for an excursion to the west coast archipelago over a weekend.

In the First Bornö Symposium the central problem to be discussed was the boundary layers of the ocean, both thermal and frictional-inertial. In addition, talks were given on stability problems, and waves, both in the atmosphere and the ocean. In the Second Borno Symposium it was suggested taking up the problems of convection and the thermal circulations, and a number of contributions in this field were given. They cover such topics as turbulent convection, salt-heat convection, influence of particles, and convection in fluids with variable coefficients. A second class of problems considered was stability of flow, especially baroclinic flows, and a third class was steady ideal fluid motion

in a rotating fluid, with special application to the ocean. By invitation, one participant presented a recent work on gravitational shockwaves with astrophysical applications.

Three evening sessions were held in the form of panel discussions with invited speakers. These covered stability theory, convection, and oceanic circulation. The contributions given in these sessions are not included in the reviews.

By invitation Prof. Louis Howard, Massachusetts Institute of Technology, gave the "Ekman-lecture" of the Symposium, in honor of the late Swedish oceanographer, V. W. Ekman.

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Woods Hole, December, 1964.

Pierre Welander.

A Recent Development in the Theory of Hydrodynamic Stability

Louis N. Howard

(Reviewed by Joseph Pedlosky)

The lecture of Louis Howard was directed toward an extension of the well-known Rayleigh condition for instability of plane parallel, incompressible, inviscid shear flow to the case of non-parallel flows. The unsatisfactory nature of the Rayleigh theorem, besides its restriction to parallel flows is that it apparently depends upon the method of solution, i.e. the Fourier analysis in time. This new theorem is not restricted to exponential solutions in the time and is independent of any periodicity condition in space as well. Briefly stated the theorem considers a steady mean flow characterized by a stream function and its associated velocity field $\vec{u}_0$ and vorticity $\Omega(\Psi_0)$. Then by a consideration of the total energy equation, correct to second order in perturbation quantities, using in addition only the constraint that the perturbation vorticity equation, Howard obtained

$$\frac{\partial}{\partial t} \int_R \left(\frac{1}{2} (|\nabla \psi|^2 - \frac{\omega^2}{\Omega'(\Psi_0)}) \right) = \int_S \frac{\vec{u}_0 \cdot \vec{n} dS \omega_{1/2}^2}{\Omega'(\Psi_0)}$$

R is the domain of the flow, S is the bounding curve while

ψ and ω are the perturbation stream function and vorticity.

The condition that $\omega^2 = 0$ where $\vec{u}_0 \cdot \vec{n} < 0$ is also placed as a requirement on the perturbation fields.

The theorem states then, that if $\Omega'(\Psi_0) < 0$ the

the perturbation amplitudes must decrease with time, which is seen by examining the equation above. A sufficient condition for stability is then $\Omega'(\Psi_0) < 0$, a simple and elegant result.

It can readily be demonstrated that this condition covers both the Rayleigh and Fjörtoft conditions for plane parallel flow.

The lecturer also pointed out the following suggestive fact. Consider the steady problem for Ψ_0 and consider the functional

$$\mathcal{F}\{\psi\} = \int_{\mathcal{R}} \{ |\nabla\psi|^2 + H(\psi_0) \}$$

where $H(\psi) = \frac{p}{\rho} + \frac{u^2}{2}$ and $H'(\psi) = -\Omega$.

The stationary points at $\mathcal{F}\{\psi\}$ correspond to the first integral of the steady vorticity equation at the stationary solution $\psi = \psi_0$

$$\Delta\psi_0 = H'(\psi_0)$$

The second variation of $\mathcal{F}$, correct to second order in the variation is

$$\delta^2\mathcal{F} = \int_{\mathcal{R}} \left[\frac{1}{2} |\nabla\psi|^2 + \frac{1}{2} H''_0(\psi_0) |\psi|^2 \right]$$

If $H''_0 = -\Omega'_0(\psi_0) > 0$ the flow is uniquely (to this order in the variation) specified by $\Omega(\psi_0)$; this condition being exactly the same as the sufficient condition for the stability of the flow. There is a hint, therefore, of a direct relationship

between the uniqueness of the steady flow and its stability. It seems intuitively reasonable that a flow with several possible equilibrium configurations for the same $\Omega(\psi_0)$ may be unstable in the sense that the fluid "wanders" from state to state. The precise connection is not however completely clear.

Remarks

George Benton made the interesting observation that if $\Omega''(\psi_0) = 0$ the theorem which was correct only for small amplitude perturbations is in fact true for the completely non-linear problem. Thus all steady flows of the form $\nabla^2 \psi_0 = k^2 \psi_0$ ($k^2 > 0$) will be stable to arbitrarily large disturbances.

An Almost Easy Stability Problem

Joseph Pedlosky

(Reviewed by Louis N. Howard)

The purpose of this investigation is to use the solutions of the Eady problem as a basis on which to construct a representation of the baroclinic stability properties of other profiles which are in an appropriate sense close to the basic Eady profile, this representation being given explicitly in terms of the profile.

The stability problem is treated as an initial-value problem, relations with the normal mode approach being investigated later. Attention being restricted to non-viscous quasi-geostrophic perturbations, the mathematical problem is reduced to the partial differential equation

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right)(\varepsilon \psi_{zz} + \psi_{xx} + \psi_{yy}) - (\varepsilon u_{zz} + u_{yy}) \psi_x = 0$$

where ψ is the stream function of the perturbation, ε a constant, and $u = \lambda z + \mu g(y, z)$ is the basic flow. λ is of order 1, μ is taken as small and it is in this sense that the problem is almost the Eady problem, which corresponds to $\mu = 0$.

Taking a Fourier transform in x and a Laplace transform in t we obtain:

$$(p + i\alpha\lambda z)(\varepsilon \varphi_{zz} + \varphi_{yy} - \alpha^2 \varphi) = i\alpha\mu(\varepsilon g_{zz} + g_{yy})\varphi - i\alpha\mu g(\varepsilon \varphi_{zz} + \varphi_{yy} - \alpha^2 \varphi) - v \quad (1)$$

The boundary conditions, corresponding to rigid boundaries at

$z=0,1$, $y=0,1$ are

$$(p+\lambda z i \alpha) \varphi_z - \lambda i \alpha \varphi = \theta - \mu g i \alpha \varphi_z + \mu g_z i \alpha \varphi$$

at $z=0,1$

and $\varphi=0$ at $y=0,1$; $v(y,z)$ and $\theta(y)$ are functions determined from the initial conditions. Without essential loss of generality one can take $\theta=0$.

For $\mu=0$ (Eady problem) the differential equation is that for the stability of plane Couette flow - the boundary conditions however are different and normal modes varying exponentially with time do exist here, there being a pair of them for each n in the finite Fourier sine transform with respect to y . The attack on eq. (1) now proceeds as follows: 1st, the finite sine transform with respect to y .

$(\varphi = \sum \varphi_m \sin m \pi y)$ gives:

$$\varphi_{mzz} - k_m^2 \varphi_m = \frac{i \alpha \mu}{p + i \alpha \lambda z} \sum A_{mn} \varphi_n - \frac{i \alpha \mu}{p + i \alpha \lambda z} \sum B_{mn} (\varphi_{nzz} - k_n^2 \varphi_n) - \frac{v_m / \varepsilon}{p + i \alpha \lambda z}$$

(2)

$$(k_m^2 = (\alpha^2 + m^2 n^2) / \varepsilon)$$

where the A 's and B 's are determined by g , and in general are functions of z . (2) is then transformed into an integral equation by multiplying it by the Green's function for $D^2 - k_m^2$ with the boundary conditions of the Eady problem on $z=0,1$.

Since g_m satisfies somewhat different boundary conditions

(differing by $O(\mu)$) the result is, on the left side of (2),

g_m plus some terms evaluated at $z = 0, 1$. On the right-hand side we get an integral over $[0, 1]$. This integral equation is

now treated by expansion in powers of μ , in a manner which is obvious except for the following: In the denominator of the Green's function for the Eady problem one finds the quadratic

$(p - p_{1m}^{(0)})(p - p_{2m}^{(0)})$ whose roots are the growth rates of the

exponential modes of the Eady problem. If expansion in μ is done in the most obvious way, one finds after Laplace inversion

that one has terms of the form $\mu t e^{p_{1m}^{(0)} t}$ as well as $e^{p_{1m}^{(0)} t}$,

i.e. something like a "secular term". While this gives the

correct answer for small μ and bounded t , it is not uni-

formly valid as $t \rightarrow \infty$. This evidently arises because the

perturbed problem has normal modes with exponential growth rates

p_{1m} , p_{2m} differing by order μ from those of the Eady

problem. To find a representation uniformly valid for large t ,

one may try to replace $p_{1m}^{(0)}$ and $p_{2m}^{(0)}$ by power series $p_{1m}(1 + \mu P'_{1m} + \dots)$

(etc.), choosing the P'_{1m} etc. to suppress the secular terms.

Actually this procedure also leads to difficulty near the point

of marginal stability of the Eady problem, but this can be avoided

by expanding the sum and product of the roots, i.e. the coeffi-

cients, of the quadratic rather than the $p_{1m}^{(0)}$ themselves. This

then is the method used. The final result is the expression of

the solution to the initial value problem as a sum of three terms,

two of which are slightly modified Eady normal modes and the third of which can be interpreted as a continuous superposition of singular modes from the continuous spectrum. The modifications of the stability characteristics are determined by the modification of the roots of the quadratic, and are explicitly given by integrals of functions determined by the initial conditions. The details of this are investigated in various special cases of interest. Another interesting aspect of the detailed investigation is the study of the behaviour of the three terms as neutral stability is approached. In this case certain singularities arise in the terms corresponding to the normal modes. These however are shown to be exactly cancelled by another singularity coming from the continuous spectrum term, and thus the singularity is only apparent from the point of view of the complete solution of the initial value problem. Actually for $t \rightarrow \infty$ the solution then reduces to a linear combination of neutral normal modes.

The Effect of Friction on Baroclinic Instability

Raymond Hide

(Reviewed by Francis P. Bretherton)

The speaker began with a brief review of previous theoretical investigations of thermal convection in the fluid annulus between two cylinders rotating together about their common vertical axis. He emphasized the difficulty of comparing the results of different workers among themselves and with experiment. Although the classical problem in baroclinic instability discussed by Eady would appear to be relevant to the transition from the upper symmetric flow regime to the steady wave regime, a discrepancy of about a factor of two remains unaccounted for. The main purpose of this study was to see whether the addition of friction at the bottom and top boundary layers is sufficient to give agreement, and the answer is "No". The results presented by Pedlosky earlier in the meeting indicate qualitatively that the horizontal shear as crudely observed would destabilize, rather than stabilize, the system.

The analysis assumes adiabatic, quasi-geostrophic, perturbations on a uniformly stably stratified basic flow which varies linearly with depth in a rectangular channel. This requires that the ratio of the vertical to horizontal temperature differences across the interior of the basic flow is comparable to the basic Rossby number based on the vertical shear and the ratio of channel depth to width. For a square channel the basic isothermal surfaces

are thus planes which make a small angle with the horizontal. Boundary layers are ignored, except that the fluctuating Ekman layers due to the perturbations cause small vertical velocities at the top and bottom of the interior region.

Substantially different criteria are obtained for the three cases in which both top and bottom boundaries are free (they will support a normal but no tangential stress), in which the bottom is rigid and the top free, and in which both are rigid. Even in the latter case the coefficient of viscosity necessary to stabilize the flow is two orders of magnitude larger than that of the experiments. The speaker concluded that the effect of the side wall boundary layers must dominate the stability criterion for the upper symmetric regime, though no quantitative theory exists. This might be possible because of their relatively large area.

No satisfactory theory exists for the basic symmetric flow in the interior region, but that observed is in qualitative agreement with that assumed here. There is, however, evidence of a narrow ageostrophic jet associated with a substantial convective heat transport across the interior. Welander asked why the flow adjusts itself so that the isotherms are nearly horizontal. The theory also ignores vertical velocities in the basic state. These are necessary if there is vertical vorticity in the interior.

The stability of the lower symmetric regime poses a difficult theoretical problem, for the perturbations cannot be geostrophic, and even ignoring viscosity the side walls prevent the governing equations from being separable.

Steady State Motion of a Perfect Fluid

George S. Benton

(reviewed by Melvin E. Stern)

The author develops the method and technique of Eulerian analysis which is quasi-Lagrangian in nature and which is useful in handling a class of non-linear fluid dynamical problems. He assumes a steady state field and neglects the molecular diffusion of heat and momentum. There is an external force (gravity) derivable from a scalar potential. The equation of state is

$$\frac{1}{\rho} = \frac{\partial G}{\partial p}(a, b, c, p)$$

where ρ is the density, p is the pressure and the differentiation is performed following a parcel whose initial coordinates were (a, b, c) .

Let us follow the motion of particles which are on the reference surface $f(x, y, z) = 0$ at time t_0 . The time interval $\tau = t - t_0$ may be taken as one of the three independent coordinates which characterize the locally steady motion. The other coordinates are stream functions (ψ, γ) obtained from the continuity equation

$$\nabla \cdot \rho \vec{v} = 0$$

$$\rho \vec{v} = \nabla \psi \times \nabla \gamma$$

Thus, for example, the Eulerian continuity equation takes the Lagrangian form

$$\frac{\partial(x, y, z)}{\partial(\psi, \gamma, \tau)} = \frac{1}{\rho},$$

and in general the substantial time derivative is given by

$$\frac{dQ}{dt} = \vec{v} \cdot \nabla Q = \left(\frac{\partial Q}{\partial \tau} \right)_{\psi, \gamma}$$

The Eulerian equations of motion have the form

$$\chi_{\tau\tau} = - \frac{\partial(p, y, z)}{\partial(\psi, \gamma, \tau)} - \frac{\partial(\phi, y, z)}{\partial(\psi, \gamma, \tau)} + \text{etc.}$$

The energy equation is formed by multiplying this by χ_ψ and adding it to the analogous equations for the other coordinates.

Thus the Bernoulli equation has the form

$$\frac{1}{2} (\chi_\tau^2 + y_\tau^2 + z_\tau^2) + G + \phi = \text{constant} = E(\psi, \gamma).$$

Two additional integrals of the motion are now required. These are formed by taking the equation for $\chi_\psi \chi_{\tau\tau}$, differentiating it with respect to τ and subtracting it from the corresponding equation formed by taking the ψ derivative of the equation for $\chi_\tau \chi_{\tau\tau}$. If $\vec{\omega}$ denotes the vorticity then these two additional equations take the form:

$$- (G)_{\psi, \gamma} \vec{\omega} \cdot \nabla \gamma = E_\psi)_{\gamma, \tau} - G_\psi)_{\gamma, p}$$

$$(G_p)_{\psi, \gamma} \vec{\omega} \cdot \nabla \psi = E_\gamma)_{\psi, \tau} - G_\gamma)_{\psi, p}$$

For two-dimensional flows one obtains equations previously obtained

by Long and others by setting $\gamma = -y$.

Motion of a Stratified Fluid
near the Equator of a Rotating Earth

Francis P. Bretherton

(Reviewed by Melvin E. Stern)

The recently discovered equatorial currents in both the atmosphere and the ocean has stimulated theoretical inquiries into the large-scale dynamics of that region. In this paper an exposition of the propagation of waves, especially the trapping effects near the equator, are given.

The author first deals with the case of the adiabatic perturbations of a highly stratified fluid, i.e. one in which the "Brunt" frequency (N) is much larger than the rotation rate (Ω). The classical approximations for a rotating sphere are made, so that we have the following system of equations for the separated components $[u, v; x, y]$ of a disturbance which (for simplicity) varies as $\sin z/H$ in the vertical direction:

$$u_t - f(y)v + p_x = 0$$

$$v_t + f(y)u + p_y = 0$$

$$\frac{p_t}{c^2} + u_x + v_y = 0$$

$$c^2 = N^2 H^2 \quad f(y) = \beta y$$

In the third equation the hydrostatic and continuity relations have been combined. (Reference: An article by Ratray in the forthcoming Hidaka Memorial Volume.) Ratray looks for eigenfunctions of the form $\exp i(kx - \omega t) \times \hat{v}(y)$. By elimination one finds the following

ordinary differential equation for the amplitude of the North-South motion

$$\frac{d^2 \hat{v}}{dy^2} + \left(\frac{\omega^2}{c^2} - \frac{k\beta}{\omega} - k^2 - \frac{\beta^2 y^2}{c^2} \right) \hat{v} = 0$$

Normalizing this equation with

$$\text{horizontal length scale } L = \sqrt{c/\beta}$$

$$\text{frequency scale } \omega = \left(\frac{1}{\beta c} \right)^{1/2}$$

gives the following:

$$\frac{d^2 \hat{v}}{d\hat{y}^2} + (2n + 1 - \hat{y}^2) \hat{v} = 0$$

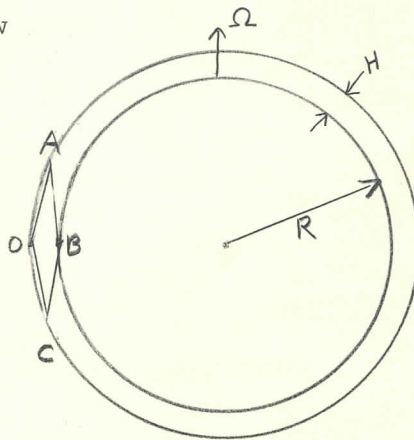
$$2n + 1 \equiv \tilde{\omega}^2 - \frac{\tilde{k}}{\tilde{\omega}} - \tilde{k}^2$$

This is Hermite's equation, whose qualitative and quantitative nature is known. A bounded analytic eigenfunction exists when n = non-negative integer, and this determines the normalized frequency $\tilde{\omega}$. The lowest order mode is $\hat{v} = e^{-\frac{1}{2}\hat{y}^2}$. This, like the other low-order modes is trapped in an equatorial zone of order L . For oceanic thermocline parameters this is of the order of 3° latitude, and periods may be as long as one week.

The detailed equation for the frequencies of oscillation is a cubic, two of whose roots are identified as inertia-gravity waves modified by the β -effect. The third and (generally) lower frequency root is identified as a Rossby wave modified by the static stability and the local coriolis force. For low-frequency oscillations the restoring force near the equator is primarily due to the β -effect. This is opposed by the effects of horizontal convergence which become increasingly important as the distance from the equator increases. Since the frequency of an inertio-gravitational wave is

always larger than the local value of the coriolis parameter, if the frequency is given, the range of latitude into which the wave can propagate is also limited.

The author then discusses the limits of validity of these solutions due to such approximations as the neglect of the vertical component of the coriolis force. To illustrate the point he considers the case of a thin shell of homogeneous water as shown in the figure below



Consider the propagation of inertia waves whose wavelength is small compared to the separation H between the two spherical shells. We may then use a ray theory to discuss the propagation of energy. The ray leaving point O is reflected at the point A so as to preserve its angle with the axis of rotation. It then propagates back towards the equator. The path $OABC$ indicates a wave which returns to its initial condition, i.e. it is trapped at the equator. The frequency of the lowest order modes is $2\Omega \sqrt{H/R}$. For this type of motion one cannot neglect the components of the coriolis force as has been done in the previous problem where the fluid was strongly stratified.

Three-Dimensional Inertial Currents in a Stratified Ocean

Allan Robinson

(Reviewed by George Benton)

Inertial currents are defined as those in which the vertical component of the relative vorticity is comparable to the Coriolis parameter. This is characteristic of certain oceanic currents, such as the Gulf Stream and the Equatorial Undercurrent. A non-viscous theory of such currents is developed and particular attention is devoted to the coupling between the regions of high current velocity and the bordering regions of geostrophic flow in the open ocean.

Let the y-axis be the direction of strong current flow. With Boussinesq approximation, and $S = 1 - \frac{\rho}{\rho_0}$, the equations of motion on a β -earth become:

$$\begin{aligned} -fv + \frac{1}{\rho_0} p_x &= 0 \\ uV_x + vV_y + wV_z + fu + \frac{1}{\rho_0} p_y &= 0 \\ -sg + \frac{1}{\rho_0} p_z &= 0 \\ u_x + v_y + w_z &= 0 \\ f &= f_0 + \beta(y \cos \theta - x \sin \theta) \end{aligned}$$

where θ is the angle between the y-axis and north. As boundary conditions, one has

$$\text{At } y=0, \quad v=v(x,0,z) \text{ and } s=s(x,0,z)$$

$$\begin{aligned} \text{At } x=\pm\infty, \quad u=U(\pm\infty,y,z), \quad v=0, \quad w=0, \quad s=S(\pm\infty,y,z) \\ \text{and } S_y = -gfV_z \end{aligned}$$

or, if there is a continental boundary at $x=c(y)$:

$$u(c(y), y, z) - v c_y = 0$$

At the bottom $z=B(x, y)$ or the top ($z=H = \text{constant}$):

$$W(x, y, B) + u B_x + v B_y = W(x, y, H) = 0$$

Let x, y, s be the independent variables and assume

s monotonic in z . The continuity equation then becomes:

$$(u z_s)_x + (v z_s)_y = 0$$

whence without loss of generality:

$$u z_s = -\psi_y$$

$$v z_s = +\psi_x$$

Also, in these coordinates, the equation for density conservation is the only one to include W . It is therefore uncoupled from the other equations. From the remaining equations it can be shown that:

$$\frac{f + v_x}{z_s} = P(\psi, s) = B_\psi$$

$$\frac{V^2}{2} + \pi = B(\psi, s)$$

where B is a form of Bernoulli function for the flow.

The equations and boundary conditions are then non-dimensional and expanded in a power series in η ($\eta = \frac{\rho_0 f_0 U_0}{\Delta \rho g H} y$, where U_0 is a velocity characteristic of $U(y, z)$ and H is the depth). If we denote by $(\xi, \eta, \theta, \zeta, \phi, \beta^*)$ the appropriate non-dimensional forms of (x, y, s, z, f) , and the down-stream variation of f , and if we set

$$\phi = \phi_0(\xi) + \beta^* \eta$$

one obtains an expansion of the potential vorticity integral (and other variables) by equating powers of η_0 in non-dimensional form:

$$V_{0\xi} + \phi_0 = \int_{0\theta} P_0$$

$$V_{1\xi} + \beta^* = \int_{0\theta} P_1 \psi_1 + \int_{0\theta} P_0$$

where each variable is expanded and

$$M = \sum M_n \eta^n$$

These equations are tractable for certain classes of flows.

Example 1. Consider a south-to-north current near a straight coast line. Let potential vorticity be a function only of density, and assume $V_0 = 0$ and $B = \text{constant}$ (flat ocean bottom). Then the first ξ -dependent approximation π_1 in the expansion

$\pi = \sum \pi_n \eta^n$ must satisfy the equation:

$$\pi_{1\xi\xi} + P_0^0(\theta) \pi_{1\theta\theta} - \frac{P_1^0}{(P_0^0)^2} \pi_1 = -\beta^*$$

where $P_N^0 = P_N^0(\theta)$. Here

$$u = -\pi_1, \quad v_1 = \pi_{1\xi}, \quad S_1 = \pi_{1\theta}, \quad \psi_1 = \pi_1 / P_0^0$$

and $\int_{0\theta} P_0^0(\theta) P_0^0(\theta) = 1.$

Consider a simply stratified fluid with constant temperature gradient. Then $P_0^0 = 1$.

Let $\pi_1 = \pi_1^\infty(\theta) + \pi(\xi, \theta)$, and $\pi = \sum a_n \pi_n(\theta) e^{-\lambda_n \xi}$

Then two equations are obtained. One in θ relates π_1^∞ , and P_I^0 .

Let $P_I^0 = +\beta^*$; let the flow vanish at the bottom; and let there be a uniform temperature at the top. The other equation then becomes:

$$\pi_1^n \lambda_m^2 + \pi_{100}^n - \beta^* \pi_1^m = -\beta^*$$

for which the eigenvalues are

$$\lambda_m^2 = \left(\frac{n\pi}{2}\right)^2 - \beta^*$$

From $n = 1$ we find $\beta^* < \left(\frac{\pi}{2}\right)^2$ is required. The inflow to the current from the open ocean is directed toward the west at all elevations. Velocity and density distributions for the first approximation can be obtained for a wide class of flows, and higher approximations can also be evaluated.

Other Examples. The conditions of no flow at the ocean bottom and of an isothermal surface are relaxed. It is shown that no eastward flow at the edge of the jet can occur. However, with bottom topography, there is a possibility of such eastward flow existing at some levels. To examine this question further, the particular case of bottom topography and a linear vertical density distribution was examined, using a power expansion in ξ (non-dimensional z) about the conditions at the boundary of the jet. In this case, also, no eastward velocities at the margin of the jet could occur.

The problem of possible existence of marginal flows away from the jet is being further examined. Also, first and higher order approximations are being computed for realistic inertial flow fields.

Convection at High Prandtl Number

Edward A. Spiegel

(Reviewed by Willem V.R. Malkus)

The search for analytical procedures which can shed light on the problems of turbulent convection was reviewed in this address. The advantage of the high Prandtl number limit is that the non-linear momentum transport terms become vanishingly important, leaving only the scalar non-linear terms, $\underline{v} \cdot \nabla T$, in the Boussinesq equations for the convection. The speaker suggested that the mean field part of these terms, βW , should be scaled in a different fashion from the fluctuating part, $(\underline{v} \cdot \nabla T - \frac{\partial \overline{WT}}{\partial z})$. When a characteristic scaling for the horizontally averaged temperature field is ΔT , it was proposed that the fluctuating temperature field may be scaled as $\Delta T/R$, where R is the Rayleigh number. In consequence, at high R and low Grasshoff number only the mean field non-linear interactions would enter into the convection problem.

Independent of the plausibility or non-plausibility of this proposal, it was felt that an exploration of solutions of the mean field equations alone would be of value in understanding the energy sources and sinks in the convection process.

The careful study of such solutions had been initiated by the speaker some years ago. He had established the general theorem that all stable solutions to the mean field problem must be time-independent. The next step was to determine the marginally-stable

eigenfunctions on an arbitrary mean temperature field. These functions are then used to determine a second mean temperature field through the relation $\beta = \frac{1}{\alpha} (H - \overline{WT})$. A solution is found when, by iteration, these two mean fields become the same.

The approach taken to determine the general eigenfunctions was an asymptotic (W.K.B.) analysis. The speaker reported on the first results of this analytic study. In the intervening years, Dr. J. Herring had been encouraged by the speaker and Dr. R. Kraichnan, to program this non-linear problem for machine computation. The exciting results from the machine have encouraged swifter progress in the more generally useful analytical method.

The stable solutions to the mean field problem resulting from the machine computations have the following properties. The heat flux is given by the correct qualitative law, but is 20% to 30% too high. The mean temperature profile boundary-layers as it must, but produces a small region of negative β not yet observed in any experiments. The mean values of W and T as a function of distance from the boundary have the qualitative features of the (limited) observations and are of the observed magnitude, but the detailed power laws differ from the observations. In total, a remarkable similarity exists between these simple, steady mean field solutions and the fully-turbulent real flow. For the first time in the study of turbulent fields, one can see a sense in which the fluctuating non-linearities can be thought of as "small" compared to other effects. This provides a hope that an adequate quantitative description of turbulent convection can be found using the "free" expansion schemes of either Kraichnan or Edwards. Either proposal,

or others which can be constructed, will cause cascading in the Fourier space, yet the accuracies of these schemes may not play a vital role in the resulting correlation fields.

Discussion of this work by Howard, Pedlosky and Bretherton shed light on the peculiar singularities which arose in the speaker's use of the asymptotic procedures. It was tentatively agreed that no practical difficulties need arise due to the existence of these singularities. After considerable philosophic exchange, it was concluded that an understanding of the dramatic success of the mean field approximation must await a third Bornó symposium.

Penetrative Convection

Willem V.R. Malkus

(Reviewed by Edward A. Spiegel)

The lecture opened with a review mentioning the mixing length and statistical theories and suggesting their inadequacies. There followed a discussion of the speaker's own approach to "MACROSCOPIC STATISTICAL PHYSICS". The discussion was illustrated by the example of turbulent convection. It was suggested that in the high Prandtl number limit the problem of turbulent convection is the simplest example of this class of problems. In this connection, work of Herring and of Spiegel was discussed.

With the foregoing discussion as background, the speaker turned to the problem of penetrating convection as exemplified by an experiment which was proposed some years ago, and which was carried out by Furumoto and Rooth. In this experiment a plane-parallel layer of water is cooled to 0°C at its lower surface and maintained at a temperature greater than 4°C at its upper surface. The point of this experiment is that in water cooler than 4°C , the density increases upward while the density in the warmer-than- 4°C decreases upward. Thus the water is locally stable or unstable when it is respectively above or below the 4° -level. The global stability of such a fluid has been studied by Veronis who also considered finite amplitude effects.

The question raised by the speaker is: What is the structure of

the transition layer between the stable and unstable fluids when the motion is turbulent?

To answer this question it is first necessary to determine the thermal structure of the lower, fully-convective fluid. This can be done by treating the lower boundary layer by the Malkus theory. The resulting heat flux then determines the temperature gradient. Extrapolation of this gradient to the point where it meets the extrapolation of the isothermal gradient fixes the location of the penetrative zone. The remainder of the discussion turned on the determination of the structure of the penetrative zone, in a self-consistent, mechanistic way. In order to completely specify the structure, the velocity of the fluid elements as they enter the penetrative zone must be specified. This velocity cannot be specified exactly, but may be upper-bounded on the basis of the speaker's theory as confirmed by his experimental studies. With the introduction of this parameter it becomes possible to estimate the vertical extent of the penetrative layer. The value reported agrees with that estimated from the experiments of Furumoto and Rooth.

A Model of Turbulent Convection
(The Ekman lecture)

Louis N. Howard

(Reviewed by George Veronis)

As a model of turbulent convection consider the initial value problem of a homogeneous (constant temperature) layer of fluid with the boundaries heated uniformly along the bottom and cooled uniformly along the top. Then at time, $t > 0$, pure conduction gives rise to a temperature distribution given by

$$T = \frac{1}{2} \Delta T \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

near the bottom boundary and a similar distribution near the top boundary. The profile is sketched in figure 1a.

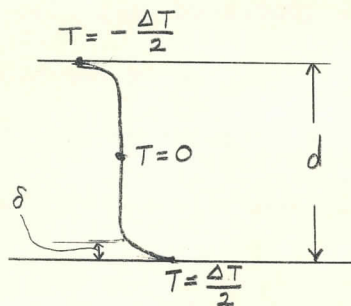


Fig. 1a

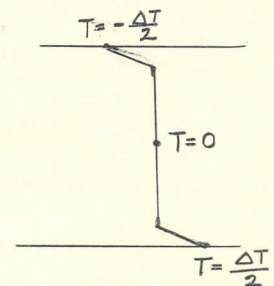


Fig. 1b

If the total depth of the layer is such that the Rayleigh number, $R = \frac{g\alpha\Delta T}{\kappa\nu} d^3$, is highly supercritical, one would expect the system to begin to convect when the depth, δ , of penetration of the boundary temperature has attained a value roughly comparable to the value

$$\delta = d_c = \left(\frac{R_c \kappa \nu}{g\alpha\Delta T}\right)^{1/3}$$

In the specific case discussed R_c was estimated as the critical

Rayleigh number for the broken line temperature profile shown in figure 1b.

As cellular-like convection begins, the effect of the instability would be to create rising plumes of warm fluid near the bottom boundary. (We confine our attention to the lower boundary. Similar arguments hold for cold plumes of fluid near the top.) These are shown in figure 2.

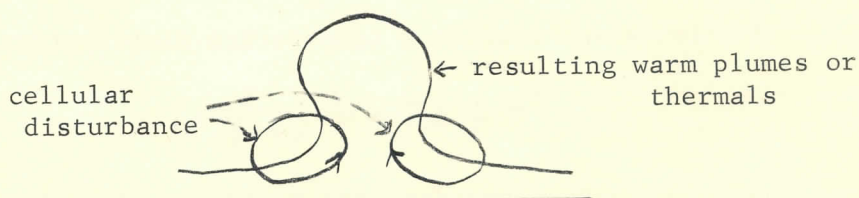


Fig. 2

Once the warm plumes have been generated they rise and since they are buoyant in the colder ambient fluid above the boundary layer, they will rise rapidly and the subsequent dynamical picture is determined by the behavior of these thermals rather than by the initially unstable cells.

The thermals are assumed to rise and to "suck out" (or be driven by) the warm fluid from the boundary layer. A warm plume thus sweeps out the neighboring region of the boundary layer and rises to the interior where it coalesces with descending cold plumes from the top boundary. A slow conductive build-up then takes place in the region where a warm plume has sucked out the warm boundary layer and the entire process, from conductive transfer into the boundary layer to convective ascent of a warm plume is

repeated. The picture implies that the warm plumes are intermittent.

It is assumed that the rising warm plumes are governed by dynamics similar to the dynamics of Taylor-Rayleigh instability of a warm layer of fluid lying below a cold layer of fluid.

The growth rate of disturbances in the latter problem is given by e^{nt} where

$$n = \left(\frac{\nu k^2 (\alpha \Delta T)^2 g^2}{k^2 \nu^2} \right)^{1/3}$$

and the horizontal wave length is given by

$$k \sim \left(\frac{g}{\nu^2} \alpha \Delta T \right)^{1/3}$$

Thus the growth rate is approximately given by

$$\left(\frac{\delta^2}{K} \right) (R_c^{-2/3}) (\sigma^{-1/3})$$

Thus for sufficiently large Prandtl no., σ , the time scale of the growth is small compared to the diffusive time scale, $\frac{\delta^2}{K}$.

Hence, the argument is plausible.

Now the heat content per unit area due to the conductive process is

$$\rho c \Delta T \sqrt{\frac{k t_*}{\pi}}$$

where t_* is the time at which a plume begins to rise. The conductive heat flux into the layer from the bottom boundary is

$$\frac{\rho c \Delta T k}{\sqrt{\pi k t_*}}$$

Since the rising plume convects all of the heat contained in the boundary layer the Nusselt number is given by the ratio of the above two expressions, viz.,

$$N = \frac{d}{\sqrt{\pi K \tau_k}} = \frac{4}{\pi} \frac{d}{\delta} = \frac{4}{\pi} \left(\frac{R}{R_c} \right)^{1/3}$$

When the value of R_c is evaluated from the profile given in figure 1b, the value of N becomes

$$N = .12 R^{1/3}$$

This can be compared with Herring's (1963) results of

$$N = .115 R^{1/3}$$

The agreement is remarkably good though the mechanistic models are physically quite different.

The motivation for the above study appears to have arisen from the observation by Townsend and others that turbulent convection is invariably accompanied by such rising warm plumes from the bottom boundary. The extremely simple analysis by Howard which results in such good agreement with Herring's extensive calculations is sufficient to warrant a careful study of the model to determine how much of the dynamics of turbulent convection is contained in such a simple picture.

Convection by Heating from Above

Pierre Welander

(Reviewed by Allan R. Robinson)

If a Boussinesq fluid is disturbed slightly from the state of hydrostatic equilibrium consistent with an imposed (uniform) vertical heat flux, the rate of release of potential energy by the ensuing motion is positive or negative definite if the heat flux is respectively parallel or anti-parallel to gravity. In the former case convection can occur.

Consider now the passage of an anti-parallel flux through an infinite medium consisting of two immiscible fluids with a level interface. Let each fluid be Boussinesq, but possess its own set of thermodynamic parameters (and hence its own Rayleigh number). Now if a small disturbance occurs of the form, e.g. for the temperature fluctuation, $\exp i(k_x x + k_y y) \Theta(z)$, where Θ is a function which vanishes sufficiently far from the interface, the potential energy release rate contains an additional term

$$\frac{1}{4} \frac{g \rho_0 \alpha \kappa}{\beta} \left(1 - \frac{\rho_0' \alpha' \kappa'}{\rho_0 \alpha \kappa} \right) \left(\Theta \frac{d\Theta}{dz} \right)_{\text{interface}}.$$

The notation is standard; primes referring to the upper layer.

Thus anti-convection can occur for sufficiently large or small value of the $\rho \alpha \kappa$ ratio if the temperature and its gradient are properly correlated.

In the exploration of the formal stability problem, exchange of stabilities is proven. The interface will remain undistorted

provided that $\nu^2 < g D^3$, where D is a typical cell dimension.

The characteristic equation which results contains the ratios of $\mu, \nu\alpha^{-1}, \chi \nu\alpha^{-1}$ in each layer and the Rayleigh number. The instability of a given wave number $K = \sqrt{K_x^2 + K_y^2}$ is shown to imply the instability of all smaller wave numbers. Sufficiently large

K values are always stable. In the limit $K \rightarrow \infty$, stability is not dependent upon the individual Rayleigh numbers, but solely by their ratios.

Although linearized theory indicates the preference for large horizontal wave-length, a finite amplitude scaling argument indicates an equilibrium amplitude proportional to K , so that some intermediate scale may be realized in nature. A feasible experiment with a water-mercury system was attempted but was not definitive. Some discussion centered around the interface conditions used and the possible effects of surface tension.

The Interaction of Large Scale Perturbations with an Equilibrium Field of Horizontally Isotropic Eddies

Melvin E. Stern

(Reviewed by Joseph Pedlosky)

The lecturer has presented a very suggestive picture of a possible interaction between a random field of turbulent two-dimensional eddies and a large-scale organized flow. The problem, although originally motivated by a consideration of the dynamics of the relatively shallow mixed region of the ocean, deserves to be considered in a more general context.

The central assertion (or assumption) of the lecturer is the following. Consider the equations for the fluctuating component of a two-dimensional velocity field, u_i , then if $\langle u_i \rangle$ denotes the ensemble mean component, the equation for the Reynolds stresses of the turbulent field is:

$$\frac{\partial}{\partial t} \langle u_i u_j \rangle + \langle u_j' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \langle u_j' u_i' \rangle \frac{\partial}{\partial x_j} \langle u_k \rangle + u_j \frac{\partial}{\partial x_j} \langle u_i' u_k' \rangle + (I_{ik} + I_{ki}) = 0$$

The terms denoted by $I_{ki} + I_{ik}$ are composed of triple correlations of the velocity field, the viscous self-dissipation of the eddies and the pressure work of the eddies. The central assertion is then made by the lecturer that $(I_{ki} + I_{ik})$ which is called the fluctuating-fluctuating interaction by the author, are negligible in this equation for the Reynolds stress and can be ignored. This is equivalent to the formal statement that the Reynolds stresses be proportional only to the mean strain of the fluid. The motivation

of this "elastic" assumption by the lecturer is obscure to the reviewer; the chief virtue appearing to be a rather brutal device to achieve closure in the turbulent dynamics.

Granted however this device, the lecturer produces theoretically the following interesting phenomena. If the turbulent field is originally unperturbed, the matrix $\langle u_i u_j \rangle$ is diagonal, i.e.

$\langle u_i u_j \rangle = E \delta_{ij}$, where E is a measure of the turbulent energy.

By considering only small disturbances produced by a large scale wave-like disturbance of non-zero mean in the $\langle u_i u_j \rangle$, a wave equation for the vorticity is obtained by coupling the above-linearized stress equation to the mean equation, of the form

$$\Omega_{\tau\tau} = E \nabla^2 \Omega$$

which will produce a traveling vorticity wave with a phase speed equal to the r.m.s. turbulent velocity. The lecturer estimates the relative magnitude of the "elastic" effect producing the wave and the Austausch mechanism by which the eddies will tend to degrade the organized wave motion. He reaches the conclusion that if the wave length of the wave is large compared to the eddy size, the wave will last a large number of wave oscillations of order wave length/eddy size.

It is difficult for the reviewer to adequately assess this theoretical picture. There are however several points that seem important. First the neglect of the triple correlations seems arbitrary in the extreme, despite the fact that it follows formally from the stress-strain law assumed (i.e., Hookian); for the formal

viewpoint only shifts the burden to an equally arbitrary assertion. It serves to close the problem but beyond the mathematical relief this offers, it does not seem justifiable. It is however a stimulating and original suggestion and should not be dismissed lightly. The system is at least consistent energetically. The problem of mechanism is not adequately understood. The small size of the eddy scale and therefore its associated dissipation lead to the conclusion that the eddy may equally well dissipate before the large scale wave completes a single cycle. The "elastic" nature of the physical system is then not clear. Finally, such motions, in the reviewer's estimation, should be observable in grid turbulence, but are not, according to the lecturer. It was suggested that the elastic mechanism may be not present in a three-dimensional turbulent field while it is present in the two-dimensional field. Formally the dynamics (as far as the lecturer considers) are identical in each case. This serves to make the elastic mechanism only more difficult to grasp.

On Vertical Convection and Diffusion in the Ocean

Bernard Saint-Guilly

(Reviewed by Pierre Welander)

In a homogeneous ocean the vertical motion can be deduced from Ekman's equations

$$-f\rho v = -p_x + \mu u_{zz}$$

$$f\rho u = -p_y + \mu v_{zz}$$

$$u_x + v_y + w_z = 0$$

with the boundary conditions $\mu u_z = \tau_{xz}^{\text{wind}}, \mu v_z = \tau_{yz}^{\text{wind}}, w = 0 (z=0)$

$$u = v = w = 0 \quad (z = -h)$$

One finds one part proportional to $\text{curl } \vec{\tau}^{\text{wind}}$, one to τ_{xz}^{wind} , and one to the east-west pressure gradient (the β -divergence).

In the presence of vertical stratification and vertical heat diffusion the problem becomes more difficult. In general it is necessary to allow for a vertical variation of the coefficients of viscosity and diffusivity. A theory is developed, based on an assumed dominance of the diffusive processes over the wind-produced processes. It leads to three independent equations for the temperature T and the stream functions ψ, χ , and π , introduced by the velocity expressions

$$u = \frac{\partial \psi}{\partial z} - \frac{1}{a \sin \Theta} \frac{\partial \chi}{\partial \varphi}, v = \frac{1}{a} \frac{\partial \chi}{\partial \Theta} - \frac{\partial \pi}{\partial z}, w = \frac{1}{a \sin \Theta} \frac{\partial \pi}{\partial \varphi} - \frac{\partial}{\partial \Theta} (\psi \sin \Theta)$$

where a is the radius of the earth, Θ is latitude and φ is longitude. An arbitrary relation between χ, ψ , and T can thus be imposed. The non-dimensionalized equations contain four characteristic numbers

$$\rho = \frac{\mu_0}{\rho_0 \alpha_0}, \quad \varepsilon = \frac{\chi_0 a}{h^2 (g h \alpha T_0)^{1/2}}, \quad \gamma = (g h \alpha T_0)^{1/2}, \quad \delta = \frac{h}{a}$$

where μ_0 and χ_0 are the coefficients of viscosity and diffusivity, respectively, h is the depth, α is the thermal expansion coefficient and T_0 the magnitude of the temperature variation. The thicknesses of the friction and thermal boundary layers are given by the relations $\rho \varepsilon \gamma \sim \frac{1}{2}$ and $\frac{\varepsilon}{\gamma} \sim \frac{1}{2}$, yielding the depth of about 25 m and 800 m, respectively.

In the second part, computation of the vertical velocity and diffusivity in the South Atlantic is presented, using nine "Meteor" stations. The vertical velocity computation is essentially the same as given by Stommel in Deep Sea Research, 1956. The mean vertical velocity over the depth 0 to 2000 m is found positive, and also from depth 2000 to 4000 m except in one case where downward velocities occur, in the subtropical convergence. Typical values are: 10^{-4} and $0.3 \cdot 10^{-4} \text{ cm s}^{-1}$ in the respective layers.

A computation is finally made of the diffusivity χ , based on an assumed vertical balance of convective and diffusive heat flux, $\omega T = \chi \frac{\partial T}{\partial z}$. Over the depth 75 m to 1000 m the diffusivity varies from about $3 \text{ cm}^2 \text{ s}^{-1}$ to $70 \text{ cm}^2 \text{ s}^{-1}$.

Of course, these methods can only serve to give rough estimates. The computation of the vertical velocity requires the knowledge of a reference level, that we do not normally have. The computation of the diffusivity neglects not only the horizontal transports into the region but neglects also any net flux in the vertical, an assumption that seems hard to justify.

Convection against a Stabilizing Salt Gradient

George Veronis

(Reviewed by S. Gösta Walin)

The author has examined the development of the convection due to heating from below on a stable salt gradient. He has thereby applied two techniques: namely, finite amplitude development around the linear solution, and time integration of a Fourier representation of the field.

Both techniques show that finite amplitude instability occurs at a much lower value of the Rayleigh number based on the temperature difference than is predicted by linear theory. Furthermore, the calculation seems to predict that the steady state is always characterized by a single cell in the vertical with salt concentration almost constant except at the horizontal boundaries where sharp gradients occur. This situation is thus quite similar to the ordinary Rayleigh convection because the salt boundary layers are much thinner than the temperature boundary layers due to the large difference in the diffusivities.

One might wonder what happens if one has a step structure in the salt concentration when the heat gradient is introduced. It is probable that in this case it would develop a system of horizontal layers (which is actually indicated by experiments), each convecting within itself, with thin stable layers in between.

However, one can imagine a physical mechanism that will turn this situation into the one presented by Veronis. Such a mechanism

would be that the thicker layers will tend to grow thicker at the cost of the thinner ones and one would end up with only one convecting layer. This development might, however, take a considerable time and the situation with steps in the salt concentration is thus to be considered an interesting quasi-steady situation which might justify some further examination.

The relation to experiments is however doubtful due to the difficulty in establishing the boundary conditions used theoretically for the salt concentration.

Stability of Water Stratified by Heat and Salt

S. Gösta Walin

(Reviewed by George Veronis)

The linear problem of a layer of fluid stratified by salt and by heat is studied for cases where the mean (linear) gradients may be either positive or negative. The study is divided into three cases: (a) the sum of the ratios of mean gradients to conductivities is negative (this applies to the salt-finger problem); (b) the sum of the products of each gradient with the sum of viscosity and its own conductivity is negative (this applies to the "layer" case found experimentally by Stommel and Turner); (c) the salt gradient vanishes (the classical Bénard problem).

By considering the dependence of the growth rate σ on the various parameters for small wave numbers, Walin shows that for cases (a) and (b) it is possible to consider systems with infinite extent in all directions. (The Bénard problem is such that one must consider the boundaries.) Essentially the conclusion is based on the fact that for small wave numbers the growth rate increases as the wave number increases. Hence, one has hopes of finding a maximum growth rate by a clever manipulation of the equations.

For each of cases (a) and (b) two criteria for growth rates as functions of wave number are determined and in each case one must choose that wave number which maximizes the growth rate. The two criteria apply to different pairs of values of the imposed gradients, one pair corresponding to near-marginal instability, the other

for more unstable systems.

For the salt finger case, Walin essentially reproduces the results of Stern. His results for the layer case are quantitatively quite close to those of the salt finger case.

Neglected finite amplitude effects will, of course, be important in determining the development of the system and one can look upon the present study as one which indicates the direction to take (or values of parameters to consider) for the finite amplitude work.

The Influence of Load Settling and Boundary Erosion
on Density Currents

Claes Rooth

(Reviewed by Francis P. Bretherton)

There were three main topics of this talk, connected, but each having wider and distinct applications. There were several interesting ideas, though few of these have been pressed far enough theoretically or experimentally for concrete conclusions to be drawn.

The first analysis concerns a suspension of equal particles falling under gravity through a fluid with terminal velocity, V , which is assumed known and unaffected by the particle concentration or the bulk fluid motion. Cooperative phenomena are thus ignored, except insofar as they arise by variations in the bulk specific gravity of the medium consisting of fluid plus particles. These variations will move relative to the medium as the particles fall through it, and may drive bulk motions. An energy argument shows that, if the fluid is incompressible, an initially uniform particle distribution is stable and no bulk motions can grow unless non-uniformities spread in from the upper boundary. Bretherton suggested that variations of terminal velocity through the local fluid, due to the presence of other particles, may cause instabilities. The horizontal interface at the base of a homogeneous cloud of particles falling into a viscous particle-free liquid is unstable, as is the interface below a liquid of larger density, although the perturbation

vorticity becomes distributed through the cloud instead of remaining concentrated at the interface. The growth rate remains bounded as the length scale of the disturbance becomes smaller.

The second topic was isolated thermals and heavy vortex rings. The experiments of Scorer, Turner and others show that a blob of heavy fluid falling through a quiescent medium forms an approximately spheroidal vortex, with the denser fluid concentrated in a ring, which entrains from behind and expands conically as it falls. A heavy vortex ring behaves similarly, and the speaker pointed out that the expansion may be simply described as the Magnus effect. Provided the circulation around a material circuit threading the centre of the vortex is conserved, for a heavy thin ring vortex or a line pair the energy gained from the gravitational forces as they impart downward impulse, all appears as kinetic energy, whereas for a spheroidal vortex, $9/14$ of this energy must be dissipated. This suggests that in the presence of slight dissipation neither a spheroidal or a thin ring vortex will persist and an intermediate state is appropriate. The characteristic velocities in the core of an expanding ring vortex increase, so if it is driven by a suspension of heavy particles in the core it will continue to support them indefinitely.

Finally came a tentative application of these ideas to turbidity currents, though much more theoretical and experimental investigation is needed. The speaker suggested that the main transport of mass in a downslope current may be in a moving concentrated vortex-like motion

at a constant distance from the slope, which maintains its circulation against friction by a large particle density not only in the core but extending somewhat behind and below it to the boundary layer on the slope. This mechanism might increase by a factor of three or so the velocities obtained on dimensional grounds for a given mass release, bringing them more in line with those observed, and would also greatly enhance velocities in the boundary layer over those appropriate for an essentially unidirectional flow. This is of importance in estimating erosion rates. His laboratory experiments with sand falling in water down an inclined metal plate lend support to this idea, though several features remain unexplained.

The Shock Wave in the Wake of a Gravitating Body
moving through a Tenuous Medium

Edward A. Spiegel

(Reviewed by Raymond Hide)

The problem of flow past a moving gravitational source is of interest in the investigation of accretion by stars and by the recently discovered collapsed massive objects, and possibly has a bearing on the dynamics of the intergalactic medium. Earlier work ignores interactions between the atoms of the gaseous medium surrounding the source. This is not a uniformly valid approximation owing to the convergence of the gravitational orbits of the atoms towards certain regions of space. The present paper takes these interactions into account and determines the structure of the shock wave behind the source when it moves with a velocity V in excess of the sound speed in the gas, C , under the assumptions that the disturbances generated are small and that the source has zero linear dimensions. This leads to a value for the drag on the gravitating object.

Owing to the idealizations employed, the paper is to be regarded as a preliminary calculation whose aim is to establish schematic approximate solutions. In common with nearly all other situations in fluid dynamics more realistic investigations are confronted by severe mathematical difficulties due to non-linear terms in the governing equations.

The equations used are those governing the motion of an ideal

fluid in a self-gravitating medium in the field of a gravitating source. Small perturbations about a stationary state are considered, attention being further restricted to phenomena having a length scale less than the "Jeans radius"

$$R_J \equiv \sqrt{\pi c^2 / 4G\rho_0}$$

where G is the universal gravitational constant, ρ_0 is the unperturbed density. The equations are solved by a Fourier transform technique, leading to one of the required results - a description of the flow field. One feature of the results is the infinite value of the perturbed density at the shock front.

In the subsonic case ($M < 1, M \equiv V/c$) the gravitational drag

$$F_1 = \frac{2\pi G^2 m^2 \rho_0}{V^2} \log_e (1 - M^2)$$

is negative. The acceleration thus produced is associated with "Jeans instability" in the medium ahead of the object. In the supersonic case, ($M > 1$) the gravitational drag

$$F_1 = -\frac{2\pi G^2 m^2 \rho_0}{V^2} \log_e \left[\left(\frac{M^2}{M^2 - 1} \right) \left(\frac{R_J}{R_0} \right)^2 \right]$$

(where m is the mass of the object and R_0 its diameter) is positive, being greater or less than the geometrical drag

$$F_2 = -\pi R_0^2 \rho_0 V^2$$

according as $1 < M < M^*$ or $M > M^*$ where M^* depends on the size of the source.

The analysis has some similarities with the theory of Berenkov radiation and with the flow of plasma around an electrically charged

satellite. A further similarity - namely with the flow produced by the slow motion of a solid body through a uniformly rotating fluid - was pointed out in the discussion.

Some discussion of the problem of including non-linear terms was also given. The paper was preceded by a brief review of astronomical work on quasi-stellar radio sources, which, owing to their enormous masses, large sizes and tremendous luminosity, represent the most important of recent scientific discoveries.